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Hisao Tominaga* Adil Yaqub†

*Okayama University
†University of California

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ON GENERALIZED n-LIKE RINGS AND RELATED RINGS

HISAO TOMINAGA and ADIL YAQUB

Throughout, R will represent a ring with (Jacobson) radical J, and N the set of all nilpotent elements in R. A ring R is called an *s-unital ring* if for each $x \in R$ there holds $x \in Rx \cap xR$. If R is an s-unital ring then for any finite subset F of R there exists an element e in R such that $ex = xe = x$ for all $x \in F$ (see, [4, Lemma 1 (a)]). Such an element e will be called a *pseudo-identity* of F. A ring R is called a *generalized n-like ring* if R satisfies the polynomial identity $(xy)^n - xy^n - x^n y + xy = 0$ for an integer $n > 1$. Recently, H. G. Moore [3] showed that if n is even or 3 then every generalized n-like ring with identity is commutative.

The present objective is to prove a theorem which generalizes Theorem 4 of [3] and deduces Theorems 2 and 3 of [3]. We begin with the following lemmas.

Lemma 1. *Suppose that for each pair of elements x, y in R there exists an integer $n = n(x,y) > 1$ such that*

$$(*) \qquad (xy)^n - xy^n - x^n y + xy = 0.$$

Then there holds the following:

(1) $(x^{n(x,x)} - x)^2 = x^{2n(x,x)} - 2x^{n(x,x)+1} + x^2 = 0$.

(2) $x^{k(n(x,x)-1)+2} = k(x^{n(x,x)+1} - x^2) + x^2$ *for all positive integers k.*

(3) *If R is semi-primitive then R is commutative.*

(4) $N^2 = 0$ and $N = J$ *contains the commutator ideal of R.*

Proof. (1) Setting $y = x$ in (*), we get (1).

(2) Let $m = n(x,x)$. Suppose $x^{k(m-1)+2} = kx^{m+1} - (k-1)x^2$. Then, by (1),

$$\begin{aligned} x^{(k+1)(m-1)+2} &= x^{m-1}x^{k(m-1)+2} = kx^{2m} - (k-1)x^{m+1} \\ &= k(2x^{m+1} - x^2) - (k-1)x^{m+1} = (k+1)x^{m+1} - kx^2, \end{aligned}$$

which completes the induction.

(3) Note that our hypothesis is inherited by all subrings and homomorphic images of R. Note also that no complete matrix ring $(S)_t$ over a division ring S $(t > 1)$ satisfies the hypothesis, as a consideration of $x = E_{11} + E_{12}$ and $y = E_{22}$ shows. Because of these facts and the structure

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theory of primitive rings, we may assume that R is a division ring. Then, since $x^{n(x,x)}-x=0$ by (1), a well-known theorem of Jacobson shows that R is commutative.

(4) Since $x^2 = x^2(2x^{n(x,x)-1}-x^{2(n(x,x)-1)})$ by (1), we see that J is a nil ideal and every nilpotent element of R squares to 0. By (3), R/J is commutative. Hence J coincides with N and contains the commutator ideal of R. Finally, if u, v are in J then $uv = uv^{n(u,v)}+u^{n(u,v)}v-(uv)^{n(u,v)} = 0$.

Lemma 2. *Let R be an s-unital ring satisfying the hypothesis in Lemma 1. Then there holds the following:*

(1) *For each $x \in R$ there exists a positive integer a such that $x^{a(n(x,x)-1)}$ is an idempotent.*

(2) *Every idempotent of R is central.*

Proof. (1) Let e be a pseudo-identity of x, and set $a = (2^{n(2e,2e)}-2)^2$. Then, by Lemma 1 (1), we get $0 = ((2e)^{n(2e,2e)}-2e)^2x = ax$. Thus, Lemma 1 (2) shows that $x^{a(n(x,x)-1)+2} = x^2$, whence (1) follows.

(2) Let a, b be idempotents in R, and e a pseudo-identity of $\{a, b\}$. According to (1), we may assume that e itself is an idempotent. We set $l = n((e-a)b,a)$ and $m = n(e-a,b)$. Then, by (*),

$$\{(e-a)b\}^l a = \{(e-a)ba\}^l-(e-a)ba^l+(e-a)ba = 0.$$

But, again by (*),

$$\{(e-a)b\}^m = (e-a)b^m+(e-a)^m b-(e-a)b = (e-a)b,$$

and therefore $\{(e-a)b\}^m a = (e-a)ba$. Reiterating in the last and using $\{(e-a)b\}^l a = 0$ above, we get $(e-a)ba=0$, and hence $ba = aba$. Replacing a by the idempotent $e-a$ in the above argument, we also have $b(e-a) = (e-a)b(e-a)$, and hence $ab = aba$. Combining these, we conclude that $ab = ba$, and thus all idempotents of R are central.

Lemma 3. (1) *R is a generalized n-like ring if and only if R satisfies the polynomial identities $(xy)^n = x^n y^n$ and $(x^n-x)(y^n-y) = 0$.*

(2) *If R is an s-unital generalized n-like ring then $(n-1)[u,x]=0$ for all $u \in N$ and $x \in R$.*

Proof. (1) If R is a generalized n-like ring, then R satisfies the polynomial identity $x^ny^n-xy^n-x^ny+xy = (x^n-x)(y^n-y) = 0$ (Lemma 1 (1) and (4)). Combining this with $(xy)^n-xy^n-x^ny+xy = 0$, we readily obtain $(xy)^n = x^ny^n$. The converse is trivial.

(2) Accordrding to Lemma 1 (4), we have

$$0=\{(xu)^n-xu^n-x^nu+xu\}-\{(ux)^n-ux^n-u^nx+ux\}=[u,x^n]-[u,x].$$

Now, let e be a pseudo-identity of $\{x, u\}$. Then, by (1) and Lemma 1 (4),

$$\begin{aligned}[u,x]=[u,x^n]&=(ux+x)^n-(xu+x)^n=\{(u+e)x\}^n-\{x(u+e)\}^n\\ &=[(u+e)^n,x^n]=n[u,x^n]=n[u,x],\end{aligned}$$

which implies (2).

We are now in a position to state our main theorem.

Theorem 1. *Let R be an s-unital (directly) indecomposable ring. Suppose that for each pair of elements x, y in R there exists an integer $n=n(x,y)>1$ such that $(xy)^n-xy^n-x^ny+xy=0$. Then R is a local ring whose characteristic is p or p^2, p a prime.*

Proof. Since R is indecomposable, Lemma 1 (4) and Lemma 2 show that R contains 1 and is a local ring. Moreover, noting that $(2^{n(2,2)}-2)^2=0$ by Lemma 1 (1), we see that the characteristic of R is a power of a prime p. Since p is in N, we get $p^2=0$ (Lemma 1 (4)).

Corollary 1. *Let R be an s-unital ring. Suppose that for each pair of elements x, y in R there exists an integer $n=n(x,y)>1$ such that $(xy)^n-xy^n-x^ny+xy=0$. Then R is a subdirect sum of local rings. If furthermore $[xy,yx]=0$ for all $x\notin N$ and $y\notin N$, then R is commutative.*

Proof. In view of Theorem 1, it remains only to prove the latter part. Note that if R^* is a homomorphic image of R then $[x^*y^*,y^*x^*]=0$ for all non-nilpotent elements x^*,y^* in R^*. Because of this fact, we may assume that R is subdirectly irreducible, and thus R is a local ring (Theorem 1). Then, noting that N is commutative (Lemma 1 (4)), we can easily see that $[xy,yx]=0$ for all x, $y\in R$. Hence,

$$[x,[x,y]]=[x(y+1),[x,y+1]]-[xy,[x,y]]=0.$$

Now, by [2, Theorem], we see that R is commutative.

Corollary 2. *Let R be an s-unital generalized n-like ring. If R is indecomposable then R is a local ring whose characteristic is p or p^2, p a prime; if p does not divide $n-1$ then R is commutative.*

Proof. In view of Theorem 1, it remains only to prove that if $(p, n-1)=1$ then R is commutative. By Lemma 3 (2), $(n-1)[u,x]=0$ for all

$u \in N$ and $x \in R$. Combining this with $p^2[u,x] = 0$, we obtain $[u,x] = 0$, and thus N is contained in the center of R. Then, using Lemma 1 (1) and [1, Theorem], we see that R is commutative.

The next includes Theorems 2 and 3 of [3].

Corollary 3. *Let R be an s-unital generalized n-like ring. If n is even or* 3, *then R is commutative.*

Proof. Without loss of generality, we may assume that R is subdirectly irreducible, and therefore R is a local ring by Theorem 1. If n is even, then $4 = \{(-1)^n - (-1)\}^2 = 0$ (Lemma 1 (1)). Hence R is commutative by Corollary 2. Next, we consider the case that $n = 3$. Since R is a local ring, it is enough to show that if x, y are units in R then $xy = yx$. By Lemma 3 (1),

$$x^2y^2 - x^2 - y^2 + 1 = x^{-1}(x^3 - x)(y^3 - y)y^{-1} = 0$$

and $y^2x^2 - y^2 - x^2 + 1 = 0$. Hence $x^2y^2 = y^2x^2$. Using this and Lemma 3 (1), we get

$$(xy)^3 = x^3y^3 = xx^2y^2y = xy^2x^2y = (xy)(yx)(xy),$$

whence it follows that $xy = yx$.

Remark. H. G. Moore required a theorem of Herstein [1] in the proof of [3, Theorem 3]. However, we can prove the same without making use of Herstein theorem (see the proof of Corollary 3).

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OKAYAMA UNIVERSITY, OKAYAMA, JAPAN
UNIVERSITY OF CALIFORNIA, SANTA BARBARA, U.S.A.

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